Stochastically stable fluctuations in a model of electrical discharge with external illumination

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(Received 9 January 1997)

A Markov description of the electron population in a model of electrical discharge between parallel plates in the presence of an external illumination source is developed. Electron production is assumed to be due to ionizing collisions in the gas, as well as photoelectric emission at the cathode; the electrons are assumed to move with a constant drift velocity towards the anode, where they are lost. The Markov description is based on a discretized distribution of electrons within the gap, from where macroscopic equations for the mean electron density and the density-density correlation function are obtained in the limit to the continuum. The results show the existence of stochastically stable solutions only when stationary discharges are obtained by means of a nonvanishing external illumination. In addition, the variance-to-mean ratio in the steady state shows a discontinuity when the conditions of the discharge are those for which the breakdown Townsend criterion is satisfied. Numerical examples are used to illustrate the results. [S1063-651X(97)10510-4]

PACS number(s): 52.80.Dy, 05.40.+j, 02.50.−r

I. INTRODUCTION

In an earlier work [1] a simple Markov description for the ionization growth in an electrical discharge between parallel plates was developed. Unlike previous related works dealing exclusively with the total electron population [2–8], a characteristic functional approach was introduced there to analyze both space- and time-stochastic fluctuations of the electron population. In that model, the ionization growth inside the gap is assumed to be due to atomic ionizing collisions and to photoemission at the cathode from deexcitation of atoms; losses in the system are accounted for by the drift motion of the electrons towards the anode due to the electric field between the plates. Diffusion processes, space-charge effects, and electron attachment and detachment are not considered. In addition, the voltage between the plates is assumed to remain constant at all times, neglecting thus the feedback effect provided by the external circuitry. Under these assumptions, it is found that the average population reaches a stationary state whenever the well-known breakdown *Townsend criterion* [2,9–11] is satisfied; however, the fluctuations around the mean behavior diverge as time evolves. The system is then stochastically unstable and it is concluded that the electron population in such discharges will eventually extinguish. When the discharges do not satisfy the Townsend criterion, both the mean electron population and its fluctuations either grow without limit (*supercritical case*) or vanish asymptotically (*subcritical case*). Hence, in the absence of any other electron sources, a stationary discharge cannot be sustained. The purpose of this paper is to extend that work, by including in the model an external source of electrons at the cathode, e.g., ultraviolet illumination, to show that a stationary discharge with finite fluctuations can be achieved under these conditions.

In most of the laboratory discharges, the external circuitry through which the discharge voltage is applied can neutralize the statistical fluctuations in the system by the effect of its Ohmic resistance, which provides self-stabilizing properties: When the current in the discharge increases, the voltage across the effective resistance of the circuit rises and the potential difference between the electrodes drops then; this in turn tends to reduce the ionization rate inside the gap and hence the current. If the external resistance can be neglected, the statistical fluctuations cannot be neutralized in this manner and the stability properties are then lost [6,12]. In practice, a breakdown cannot be sustained when the applied voltage between the electrodes is exactly the Townsend voltage; it is necessary that a small overvoltage exists, ensuring an expanding reproduction of the electron population [12] and, in the absence of any electric-field distortion due to space-charge accumulation, a stationary discharge is obtained due to the feedback effect just described.

A Markov description of population dynamics requires homogeneity conditions in the system under consideration [1,13,14]. Thus any localized fluctuation in a homogeneous system spreads out instantaneously, as compared to the characteristic time of the system, so that global microscopic quantities depend only on the values they assumed at a previous time. An electrical discharge does not constitute a homogeneous system since the mere information of the total population at a given time does not determine completely its future evolution. Fluctuations in the total electron population are thus local phenomena and can only be described by introducing the spatial density as a random object. Hence a Markov description of the evolution of an electrical discharge can be constructed only by taking into account both the space and the time distribution of the electron population.

The model adopted here consists of a simple one-dimensional description of an electrical discharge between parallel plates, which extends from $x=0$ to L. There exist a uniform electric field $\mathcal{E}$ between the electrodes and a filling gas at pressure $\mathcal{P}$, with $e\mathcal{E}/\sigma\mathcal{P}\ll 1$, where e is the electronic charge and σ the cross section for momentum transfer [1,8]. Although this system is not homogeneous, local homogeneity conditions are introduced by dividing the configuration space into a certain number K of cells of width Δx, in which those conditions prevail. The distribution of electrons along the cells is assumed to change due to the following processes: (i) an electron may suffer an ionizing collision with an atom in the gas, producing an additional electron at the same position, with a probability per unit time a; (ii) an electron may excite an atom to emit a photon after a negligible delay, producing a photoelectron emitted from the cathode, and the entire process is here represented by a probability per unit time b; (iii) a photoelectron may also be produced at the cathode due to the incidence of external illumination, with a probability per unit time $I(t)$; and finally, (iv) an electron within a cell may move into an adjoining cell with a probability per unit time $\mu/\Delta x$, where μ is a constant drift velocity. The voltage across the gap is assumed constant at all times and thus, similarly to the previous model, no feedback effect due to the external circuitry is taken into account. In addition, diffusion processes, space-charge effects, as well as electron attachment and detachment are also neglected here.

Within the restrictions given above, in Sec. II we construct a master equation for the conditional probability $P(\{n_\lambda\},t|\{n_\lambda^{(0)}\},0)$ of having a distribution of electrons $\{n_\lambda\}$ at a time t given an initial distribution $\{n_\lambda^{(0)}\}$ at $t=0$, with n_λ denoting the number of electrons in the λth cell. Taking the limit when the number of cells goes to infinity and their width Δx goes to zero, in the dynamical equation for the characteristic function associated with P, an equation for the characteristic functional of the continuous electron distribution $n(x,t)$ is obtained. First- and second-order functional derivatives of this latter equation yield evolution equations for the mean electron density and the density-density correlation function, respectively. The general solution to the equation for the mean electron density, as well as the associated mean total population, is obtained in Sec. III. In Sec. IV the solution to the equation for the density-density correlation function is analyzed in general and it is shown that stationary solutions with finite fluctuations exist only for subcritical discharges in the presence of an external illumination source; this particular case is studied in detail. Numerical examples are used to illustrate the results. Finally, Sec. V contains some concluding remarks.

II. EVOLUTION EQUATIONS FOR THE MEAN ELECTRON DENSITY AND THE DENSITY-DENSITY CORRELATION FUNCTION

As mentioned above, the spatial distribution of electrons along the gap is accounted for by discretizing the configuration space in a number K of cells of width $\Delta x = L/K$. Thus the distribution of electrons within the cells is given by the time-dependent random state vector $\mathbf{n}\equiv\{n_1,n_2,\ldots,n_K\}$, constituting a Markov process whose statistical properties are described by the conditional probability $P(\mathbf{n},t|\mathbf{n}^{(0)},0)$. The evolution of the conditional probability P satisfies a master equation of the form

$$\frac{d}{dt}P(\mathbf{n},t|\mathbf{n}^{(0)},0)=\sum_{\mathbf{m}}\{Q(\mathbf{m}|\mathbf{n})P(\mathbf{m},t|\mathbf{n}^{(0)},0)-Q(\mathbf{n}|\mathbf{m})P(\mathbf{n},t|\mathbf{n}^{(0)},0)\}, \tag{1}$$

where $Q(\mathbf{m}|\mathbf{n})$ is the transition probability per unit time from a distribution $\mathbf{m}$ to a distribution $\mathbf{n}$. The conditional probability P must satisfy the initial condition

$$P(\mathbf{n},0|\mathbf{n}^{(0)},0)=\delta_{n_1,n_1^{(0)}}\delta_{n_2,n_2^{(0)}}\cdots\delta_{n_K,n_K^{(0)}}, \tag{2}$$

as well as the normalization condition

$$\sum_{\mathbf{n}}P(\mathbf{n},t|\mathbf{n}^{(0)},0)=1, \tag{3}$$

for all $t\geqslant 0$.

Electron gain and losses in each cell are accounted for in the model through the probabilities a, b, $I(t)$, and $\mu/\Delta x$, as set in the Introduction. The transition probability per unit time in this case is given by

$$Q(\mathbf{m}|\mathbf{n})=a\sum_{j=1}^{K}m_j[\delta_{n_1,m_1}\delta_{n_2,m_2}\cdots\delta_{n_K,m_K}]_{m_j\to m_j+1}+b\left(\sum_{j=1}^{K}m_j\right)\delta_{n_1,m_1+1}\delta_{n_2,m_2}\cdots\delta_{n_K,m_K}+I(t)\delta_{n_1,m_1+1}\delta_{n_2,m_2}\cdots\delta_{n_K,m_K}$$
$$+\frac{\mu}{\Delta x}\sum_{j=1}^{K-1}m_j[\delta_{n_1,m_1}\delta_{n_2,m_2}\cdots\delta_{n_K,m_K}]_{m_j\to m_j-1,m_{j+1}\to m_{j+1}+1}+\frac{\mu}{\Delta x}m_K\delta_{n_1,m_1}\delta_{n_2,m_2}\cdots\delta_{n_K,m_K-1}, \tag{4}$$

where the notation $m_j\to m_j\pm 1$ indicates replacement of m_j by $m_j\pm 1$ in the corresponding Kronecker δ for each value of j in the summation. The transition probability per unit time in Eq. (4) can be obtained from the corresponding expression in Ref. [1] by adding to the total electron source at the cathode $b\Sigma_j m_j$ the source $I(t)$ due to external illumination.

The ionization growth in the discharge gap is adequately described by the Markov process specified in Eqs. (1) and (4) only in the limit when the size of the cells tends to zero, so that the homogeneity conditions within each cell are satisfied. In this limit, a convenient alternative representation of the conditional probability P is that of the characteristic functional [13,15]

$$C[\theta(x);t]\equiv\left\langle \exp\left(i\int_0^L dx'\,n(x')\,\theta(x')\right)\right\rangle, \tag{5}$$

where $\theta(x)$ is a continuous function, conjugate to the microscopic density, $n(x_j)\equiv\lim_{\Delta x\to 0}\{n_j/\Delta x\}$ [1]. It can be shown then that the characteristic functional $C[\theta;t]$ satisfies the partial integro-differential equation

$$-i\frac{\partial}{\partial t}C[\theta;t] = \int_0^L dx\{a(1-e^{i\theta(x)})+b(1-e^{i\theta(0)})\}\frac{\delta C[\theta;t]}{\delta\theta(x)dx} + iI(t)(1-e^{i\theta(0)})C[\theta;t] + i\mu\int_0^L dx\ \theta(x)\frac{\partial}{\partial x}\left(\frac{\delta C[\theta;t]}{\delta\theta(x)dx}\right) + i\mu\left[\theta(x)\frac{\delta C[\theta;t]}{\delta\theta(x)dx}\right]_{x=0}, \tag{6}$$

where $\theta(L^+)\equiv 0$ and

$$\frac{\delta C[\theta(x);t]}{\delta\theta(x')dx'}\equiv\lim_{\epsilon\to 0}\frac{d}{d\epsilon}C[\theta(x)+\epsilon\delta(x-x');t] \tag{7}$$

is the functional derivative of C [16].

The mean electron density and the density-density correlation function are obtained, respectively, by taking the first- and second-order functional derivatives of the characteristic functional in Eq. (5), that is,

$$\left[\frac{\delta C[\theta;t]}{\delta\theta(x')dx'}\right]_{\theta=0}=i\langle n(x',t)\rangle, \qquad \left[\frac{\delta^2 C[\theta;t]}{\delta\theta(x')dx'\,\delta\theta(x'')dx''}\right]_{\theta=0}=-\langle n(x',t)n(x'',t)\rangle. \tag{8}$$

The average $\langle n(x,t)\rangle$ represents the mean density of electrons as a function of the position x in the gap at a time t, whereas $\langle n(x',t)n(x'',t)\rangle$ accounts for statistical correlations between the microscopic densities of electrons at two different positions x' and x'' at the same time t.

From Eqs. (6) and (8) the mean electron density and the density-density correlation function are found to satisfy, respectively, the partial integro-differential equations

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}\right)\rho(x,t)=a\rho(x,t)+b\,\delta(x)\int_0^L dx'\rho(x',t) + I(t)\,\delta(x)-\mu\,\delta(x)\rho(x,t) \tag{9}$$

and

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x'}+\mu\frac{\partial}{\partial x''}\right)R(x',x'';t) = 2aR(x',x'';t)+a\,\delta(x'-x'')\rho(x',t) + I(t)\,\delta(x')\,\delta(x'')+b\,\delta(x')\int_0^L dx\ R(x,x'';t) + b\,\delta(x'')\int_0^L dx\ R(x',x;t)+b\,\delta(x')\,\delta(x'') \times\int_0^L dx\ \rho(x,t)-\mu\,\delta(x')R(x',x'';t) - \mu\,\delta(x'')R(x',x'';t)+\mu\,\delta(x'-L)\,\delta(x''-L)\rho(x',t), \tag{10}$$

in which the following definitions have been used:

$$\rho(x,t)\equiv\langle n(x,t)\rangle \tag{11}$$

for the mean electron density and

$$R(x',x'';t)\equiv\langle\delta n(x',t)\,\delta n(x'',t)\rangle \tag{12}$$

for the density-density correlation function; here $\delta n(x,t)\equiv n(x,t)-\langle n(x,t)\rangle$ denotes local-density fluctuations at a position x and a given t. The left-hand side of Eq. (9) represents the material derivative of the mean electron density ρ. The first term on the right-hand side of Eq. (9) is the source due to primary electron production, i.e., ionizing collisions; the next two terms represent the secondary production at the cathode, due to excitation collisions inside the gap and external illumination, respectively; the last term is a negative boundary source of electrons at the cathode, whose role is to discard any contribution to the electron population coming from the left-hand side of the cathode since this is physically not possible. A similar interpretation holds for Eq. (10).

The mean total population of the electrons in the gap $N(t)$ and the fluctuations around this value $\sigma(t)$ are obtained from the mean density and the density-density correlation function as

$$N(t)=\int_0^L dx\ \rho(x,t) \tag{13}$$

and

$$\sigma^2(t)=\int_0^L dx\int_0^L dx'\,R(x,x';t), \tag{14}$$

where $\sigma^2(t)\equiv\langle\delta N^2(t)\rangle$ is the variance of the statistical distribution of the electron population at time t. In the following section, we obtain the solution to Eq. (9) for the mean electron density. The variance of the total electron population is analyzed in Sec. IV.

III. MEAN ELECTRON POPULATION

By performing a Laplace transform [17] for the time domain, Eq. (9) can be cast into an ordinary differential equation for $\tilde{\rho}(x,s)$, the mean density in the s domain. The solution to this equation in the interval $0<x<L$ is given by

$$\tilde{\rho}(x,s)=\frac{1}{\mu}\int_0^x dx'\,\rho(x',0)e^{-(s-a)(x-x')/\mu}+\frac{1}{\mu}\{b\tilde{N}(s)+\tilde{I}(s)\}e^{-(s-a)x/\mu}, \tag{15}$$

where $\rho(x,0)$ is the mean electron density at time zero and $\tilde{N}(s)$ and $\tilde{I}(s)$ are the Laplace transforms of the mean total population (13) and the probability $I(t)$ of electron production due to external illumination, respectively. Following a method similar to that in Ref. [1], $\tilde{N}(s)$ can be shown to be given by

$$\tilde{N}(s)=\frac{N(0)-\int_0^L dx'\,\rho(x',0)e^{-(s-a)(L-x')/\mu}+\tilde{I}(s)[1-e^{-(s-a)L/\mu}]}{(s-a-b)\left[1+\frac{b}{s-a-b}\,e^{-(s-a)L/\mu}\right]}, \tag{16}$$

where $N(0)$ denotes the total electron population at $t=0$. The inverse Laplace transform of the above expressions can be easily found by substituting in Eqs. (15) and (16) the expansion

$$\left[1+\frac{b}{s-a-b}\,e^{-(s-a)L/\mu}\right]^{-1}=\sum_{n=0}^{\infty}\frac{(-b)^n}{(s-a-b)^n}\,e^{naL/\mu}e^{-nLs/\mu}. \tag{17}$$

It follows that the mean electron density in the interval $(0,L)$ is determined by

$$\begin{aligned}\rho(x,t)={}&e^{at}\rho(x-\mu t,0)\,\theta(x-\mu t)+\frac{b}{\mu}\,e^{at}N(0)\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\,e^{b[t-(x+nL)/\mu]}[t-(x+nL)/\mu]^n\,\theta(t-(x+nL)/\mu)\\&-\frac{b}{\mu}\,e^{at}\int_0^L dx'\,\rho(x',0)\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}e^{b\{t-[x-x'+(n+1)L]/\mu\}}\{t-[x-x'+(n+1)L]/\mu\}^n\,\theta(t-[x-x'+(n+1)L]/\mu)\\&+\frac{1}{\mu}\,e^{ax/\mu}I(t-x/\mu)\,\theta(t-x/\mu)+\frac{b}{\mu}\,e^{-bx/\mu}\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\,e^{-nbL/\mu}\left\{\int_0^t d\tau\,I(t-\tau)e^{(a+b)\tau}[\tau-(x+nL)/\mu]^n\right.\\&\left.\times\theta(\tau-(x+nL)/\mu)-e^{-bL/\mu}\int_0^t d\tau\,I(t-\tau)e^{(a+b)\tau}\{\tau-[x+(n+1)L]/\mu\}^n\,\theta(\tau-[x+(n+1)L]/\mu)\right\}\end{aligned} \tag{18}$$

and the mean total population is given by

$$\begin{aligned}N(t)={}&e^{(a+b)t}N(0)\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\,e^{-nbL/\mu}[t-nL/\mu]^n\,\theta(t-nL/\mu)-e^{(a+b)t}\int_0^L dx'\,\rho(x',0)e^{-bx'/\mu}\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\\&\times e^{-b(n+1)L/\mu}\{t-[(n+1)L-x']/\mu\}^n\,\theta(t-[(n+1)L-x']/\mu)+\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\,e^{-nbL/\mu}\\&\times\int_0^t d\tau\,I(t-\tau)e^{(a+b)\tau}[\tau-nL/\mu]^n\,\theta(\tau-nL/\mu)-\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\,e^{-(n+1)bL/\mu}\int_0^t d\tau\,I(t-\tau)e^{(a+b)\tau}\\&\times[\tau-(n+1)L/\mu]^n\,\theta(\tau-(n+1)L/\mu),\end{aligned} \tag{19}$$

where $\theta(x)$ is the Heaviside step function defined as

$$\theta(x)=\begin{cases}0, & x\leqslant 0\\ 1, & x>0.\end{cases} \tag{20}$$

Equations (18) and (19) are analytical expressions to evaluate dynamical macroscopic properties in the system, such as the electronic current at the cathode. The long-time limit of these quantities, if exists, is obtained from [17]

$$\rho_\infty(x)=\lim_{s\to 0}\{s\tilde{\rho}(x,s)\},\qquad N_\infty=\lim_{s\to 0}\{s\tilde{N}(s)\}. \tag{21}$$

Thus, taking this limit in Eq. (15), we get

$$\rho_\infty(x)=\frac{e^{ax/\mu}}{\mu}\{bN_\infty+I_\infty\}, \tag{22}$$

where I_∞ is the asymptotic value of the external illumination. Integrating Eq. (22) over the gap length, we obtain the relation

$$N_\infty\left\{1-\frac{b}{a}(e^{aL/\mu}-1)\right\}=I_\infty\frac{1}{a}(e^{aL/\mu}-1), \tag{23}$$

from which two different physical situations for the existence of a long-time finite electron population arise.

(i) When the parameters of the discharge satisfy the condition

$$b=b_{\rm crit}\equiv\frac{a}{e^{aL/\mu}-1}, \tag{24}$$

known as the *critical condition* or Townsend breakdown criterion [2,9–11], it follows from Eq. (23) that a finite value of N_∞ exists only when $I_\infty=0$. In this case, Eq. (16) for the mean total electron population, together with Eq. (21), yields

$$N_\infty=\frac{\left[N(0)-\int_0^L dx\,\rho(x,0)e^{a(L-x)/\mu}\right]+\tilde{I}(s=0)[1-e^{aL/\mu}]}{1-\frac{bL}{\mu}e^{aL/\mu}}, \tag{25}$$

with a mean electron density given by

$$\rho_\infty(x)=\frac{b}{\mu}N_\infty e^{ax/\mu}. \tag{26}$$

(ii) When the critical condition, i.e., Eq. (24), is not satisfied, a finite positive value of N_∞ in Eq. (23) implies $I_\infty=I_0$, with I_0 a positive constant, from which the condition

$$\frac{b}{a}(e^{aL/\mu}-1)<1 \tag{27}$$

follows, i.e., $0\leqslant b<b_{\rm crit}$, referred to as the *subcritical condition*. In this case, the mean total electron population, obtained from Eq. (23), is

$$N_\infty=I_0\frac{e^{aL/\mu}-1}{a-b(e^{aL/\mu}-1)}, \tag{28}$$

while the corresponding mean electron density, obtained from this equation and Eq. (22), is

$$\rho_\infty(x)=\frac{I_0e^{ax/\mu}}{\mu}\left[\frac{a}{a-b(e^{aL/\mu}-1)}\right]. \tag{29}$$

The stationarity condition in Eq. (28) can be deduced from physical arguments as follows. Let us assume that, at an arbitrary time t_0, a total population of $N(t_0)$ electrons exists within the gap. As they travel towards the anode, the population of these electrons will increase due to direct ionization collisions and, at the same time, will decrease as electrons reach the anode. Thus, after one transit time L/μ, i.e., at time t_0+L/μ, the original electrons and their primary descendants will have disappeared. On the other hand, at any intermediate time $t_0<t<t_0+L/\mu$, secondary electrons are being produced at the cathode at a rate $bN(t)+I_0$. Hence, after one transit time, the total number of electrons produced by them is $\int_0^{L/\mu}d\tau[bN(\tau+t_0)+I_0]\exp[a(L/\mu-\tau)]$. The stationary condition implies that this total number of electrons must be equal to the initial population $N(t_0)$ and be independent of the particular time t_0, i.e., $N(t_0)=N(\tau+t_0)=N_\infty$, from which Eq. (28) follows. The stationary condition can be written as

$$\left(\frac{bN_\infty+I_0}{aN_\infty}\right)(e^{aL/\mu}-1)=1, \tag{30}$$

which is a generalization of the previous stationarity criterion in the absence of external sources, where the ratio $(bN_\infty+I_0)/aN_\infty$ is now interpreted as the total number of secondary electrons produced by primary electron [9–11].

In order to illustrate these results, we will analyze two cases in which there are no electrons present at time $t=0$, i.e., $N(0)=0$, and where an external illumination source is impinging on the surface of the cathode. To simplify the notation, hereinafter we introduce the dimensionless quantities $\mu t/L\rightarrow t$, $x/L\rightarrow x$, $aL/\mu\rightarrow a$, $bL/\mu\rightarrow b$, and $IL/\mu\rightarrow I$ to denote time, position, and probabilistic coefficients, respectively. In both cases, we take $a=2$, so that $b_{\rm crit}=0.313\,04$, and show the time behavior of the mean population $N(t)$ as predicted by Eq. (19), for different values of the parameter b.

In the first example, the illumination is constant and equal to I_0 during the time interval $0\leqslant t\leqslant\tau\equiv1.5$ and zero afterward. Figure 1 shows the resulting behavior of the average total population, where fast growth during the first transit time $(t\leqslant1)$ is observed due to the fact that there are only electron sources, but no losses. When the initial electrons

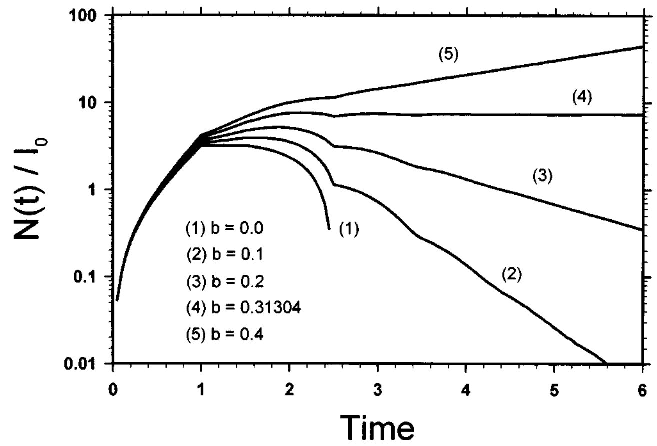

FIG. 1. Time behavior of the average total-electron population for a time-dependent illumination source of the form $I(t)=I_0$ for $0<t\leqslant1.5$ and zero otherwise. The plots correspond to discharges with the same primary production parameter $a=2$, for several values of the secondary production parameter b, satisfying (1)–(3) subcritical, (4) critical ($b_{\rm crit}=0.313\,04$), and (5) supercritical conditions. Dimensionless quantities are used, as defined in the text.

reach the anode at $t=1$, the losses become important: For a negligible secondary electron production $b=0$, the population reaches a steady value at this time and remains there until the external illumination at the cathode is removed at time $t=1.5$; for times $t>1.5$, it decays steadily until it completely disappears at $t=2.5$. A similar behavior at the early stages occurs for discharges with $0<b<b_{\rm crit}$, although, in contrast to the case with $b=0$, the population still grows during the intermediate times, but at a slower pace; after the illumination source is removed, at $t=1.5$, the population eventually decreases and asymptotically disappears. When $b=b_{\rm crit}$, the population reaches a stationary finite value, given by $I_0\tau(1-e^a)(1-b_{\rm crit}e^a)^{-1}$, as follows from Eq. (25). Finally, when $b>b_{\rm crit}$ the population grows without limit.

In the second example, we present the resulting behavior of the total electron population for a time-independent illumination I_0 for the same discharge conditions as in Fig. 1. As observed in Fig. 2, for $0\leqslant b<b_{\rm crit}$, the population reaches an asymptotic steady-state value, given by Eq. (28), within a few transit times. For $b\geqslant b_{\rm crit}$, the population increases monotonically, without reaching a stationary value, and, for critical discharges, it approaches asymptotically a linear behavior at long times (see Appendix A); thus, in particular, for $a=2$ and $b=0.313\,04$, i.e., $b_{\rm crit}$,

$$\frac{N(t)}{I_0}\rightarrow 4.866t-1.342. \tag{31}$$

IV. FLUCTUATIONS AROUND THE MEAN ELECTRON POPULATION

Statistical correlations between the local-density fluctuations at two different positions are accounted for by the density-density correlation function $R(x',x'';t)$, which satisfies the partial integro-differential equation given by Eq. (10). This equation can be solved formally by means of a double Laplace transform for the space domain, yielding

$$\begin{aligned}R(x',x'';t)=&e^{2at}R(x'-\mu t,x''-\mu t;0)\,\theta(x'-\mu t)\,\theta(x''-\mu t)\\&+\delta(x'-x'')\rho(x',t)\{(e^{at}-1)\,\theta(x'-\mu t)\\&+(e^{ax'/\mu}-1)\,\theta(\mu t-x')\}+\frac{1}{\mu}\delta(x'-x'')\\&\times e^{2ax'/\mu}\{bN(t-x'/\mu)\\&+I(t-x'/\mu)\}\theta(\mu t-x')\\&+\frac{b}{\mu}e^{2ax'/\mu}F(x'-x'',t-x''/\mu)\\&\times\theta(x'-x'')\,\theta(t-x''/\mu)\\&+\frac{b}{\mu}e^{2ax'/\mu}F(x''-x',t-x'/\mu)\\&\times\theta(x''-x')\,\theta(t-x'/\mu),\end{aligned} \tag{32}$$

where the following definition has been introduced:

$$F(x,t)\equiv\int_0^L dx'R(x,x';t). \tag{33}$$

It follows from Eq. (12) that $F(x,t)$ represents the correlation between the local fluctuations of the electron density at a position x in the gap and the fluctuations of the total electron population, at the same time t, $\langle\delta n(x,t)\delta N(t)\rangle$. According to the definition of $F(x,t)$, both sides of Eq. (32) can be integrated over the gap length to obtain a closed integral equation for $F(x,t)$, namely,

$$\begin{aligned}F(x,t)=&e^{2at}\theta(L-\mu t)\,\theta(x-\mu t)\int_0^{L-\mu t}dx'R(x-\mu t,x';0)\\&+\rho(x,t)\{(e^{at}-1)\,\theta(x-\mu t)+(e^{ax/\mu}-1)\\&\times\theta(\mu t-x)\}+\frac{1}{\mu}e^{2ax/\mu}[bN(t-x/\mu)\\&+I(t-x/\mu)]\,\theta(t-x/\mu)+\frac{b}{\mu}e^{2ax/\mu}\theta(t-x/\mu)\\&\times\left\{\int_0^x dx'e^{-2ax'/\mu}F\left(x',t-\frac{x-x'}{\mu}\right)\right.\\&\left.+\int_0^{L-x}dx'F(x',t-x/\mu)\right\}+\frac{b}{\mu}e^{2ax/\mu}\\&\times\theta(x-\mu t)\int_{x-\mu t}^{x}dx'e^{-2ax'/\mu}F\left(x',t-\frac{x-x'}{\mu}\right).\end{aligned} \tag{34}$$

A general time-dependent solution to this equation is not available. However, as shown below, an exact finite stationary solution can always be found for subcritical discharges as long as the external illumination at the cathode does not vanish at long times.

It follows from Eq. (34) that the long-time limit of the function $F(x,t)$, when it exists, must satisfy

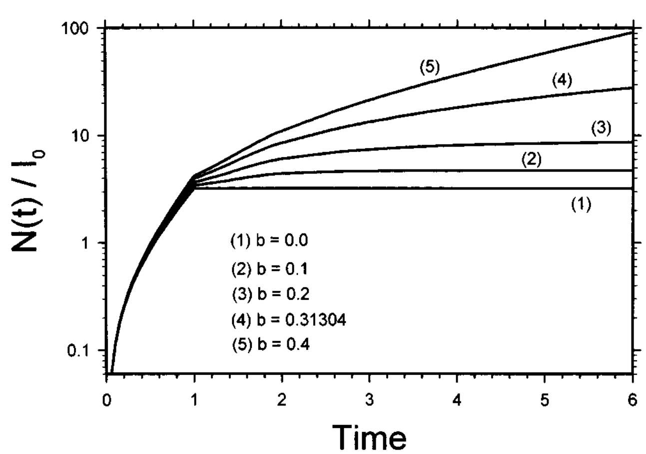

FIG. 2. Time behavior of the average total-electron population for a constant illumination source $I(t)=I_0$. The plots illustrate the average population for the same discharge parameters as in Fig. 1.

$$F(x)=(e^{ax/\mu}-1)\rho_\infty(x)+\frac{1}{\mu}\{bN_\infty+I_\infty\}e^{2ax/\mu}+\frac{b}{\mu}e^{2ax/\mu}\left\{\int_0^x dx'e^{-2ax'/\mu}F(x')+\int_0^{L-x}dx'F(x')\right\}, \tag{35}$$

where $F(x)\equiv\lim_{t\to\infty}F(x,t)$. This equation can be cast into the following ordinary nonhomogeneous differential equation with constant coefficients:

$$\frac{d^2}{dx^2}F(x)-\frac{2a}{\mu}\frac{d}{dx}F(x)+\frac{a^2}{\mu^2}(1-\kappa^2)F(x)=q(x), \tag{36}$$

with

$$q(x)\equiv[bN_\infty+I_\infty]\left\{\frac{a^2}{\mu^2}+\frac{ab}{\mu^2}+\frac{ab}{\mu^2}e^{aL/\mu}\right\}e^{ax/\mu} \tag{37}$$

and

$$\kappa^2\equiv1-\frac{b^2}{a^2}\left[e^{2aL/\mu}-1-\frac{2a}{b}\right]. \tag{38}$$

We shall point out that $1\geqslant\kappa^2>0$ for subcritical discharges, i.e., when Eq. (27) is satisfied; $\kappa^2=0$ when the critical condition in Eq. (24) is satisfied; and $\kappa^2<0$ for supercritical discharges.

In the case of subcritical discharges ($I_\infty=I_0>0$), it is easy to verify that there exists a finite stationary solution satisfying both Eqs. (35) and (36), given by

$$F(x)=e^{ax/\mu}(C_1e^{a\kappa x/\mu}+C_2e^{-a\kappa x/\mu}+C_3), \tag{39}$$

where

$$C_1\equiv\frac{1}{\mu}\{bN_\infty+I_0\}\left(\frac{\kappa+1}{\kappa}\right)\frac{ae^{(1-\kappa)aL/\mu}}{a-b\left(\frac{e^{(1-\kappa)aL/\mu}-1}{1-\kappa}\right)}, \tag{40}$$

$$C_2\equiv\frac{1}{\mu}\{bN_\infty+I_0\}\left(\frac{a}{b}\right)\left(\frac{1-\kappa^2}{\kappa}\right)\frac{a+b/(1-\kappa)}{a-b\left(\frac{e^{(1-\kappa)aL/\mu}-1}{1-\kappa}\right)}, \tag{41}$$

and

$$C_3\equiv-\frac{1}{\mu}\{bN_\infty+I_0\}\left(\frac{1}{\kappa^2}\right)\left[1+\frac{b}{a}(1+e^{aL/\mu})\right], \tag{42}$$

where N_∞ is the asymptotic value of the mean electron population, obtained from Eq. (28). Finally, from Eqs. (14) and (33), the variance at long times is obtained by simply integrating Eq. (39) over the gap length, yielding

$$\sigma_\infty^2=\frac{\mu}{a}\left\{C_1\frac{e^{(1+\kappa)aL/\mu}-1}{1+\kappa}+C_2\frac{e^{(1-\kappa)aL/\mu}-1}{1-\kappa}+C_3(e^{aL/\mu}-1)\right\}. \tag{43}$$

This expression can be shown to be a finite, positive quantity only in the interval $1\geqslant|\kappa|>0$. When $\kappa\to0$ the electron density fluctuations are such that $F(x,t)\to\infty$ as $t\to\infty$. This latter result is in agreement with our previous conclusion regarding critical discharges with no external illumination that, even though the mean population reaches a stationary value, the fluctuations grow in time without limit [1]. Finally, in supercritical cases both the mean population and its fluctuations diverge with time. We conclude that only in the case of subcritical discharges there exists a stationary state, which is characterized by Eqs. (28) and (43) for the mean total electron population and its fluctuations, respectively.

Based on the above discussion, we now look for a time-dependent solution in the subcritical case, which asymptotically approaches the final stationary state, Eqs. (39) and (43), at long times. To this end, we introduce a dimensionless smallness parameter

$$\epsilon\equiv bL/\mu \tag{44}$$

and assume that the initial electron distribution is deterministic, i.e., $R(x',x'';0)=0$, and therefore $F(x,0)=0$. Using this parameter, Eq. (34) can be rewritten as

$$\begin{aligned}F(x,t)=&\rho(x,t)\{(e^{at}-1)\theta(x-\mu t)+(e^{ax/\mu}-1)\theta(\mu t-x)\}\\&+\frac{1}{\mu}e^{2ax/\mu}I(t-x/\mu)\theta(t-x/\mu)+\epsilon e^{2ax/\mu}L^{-1}\\&\times\left\{N(t-x/\mu)\theta(t-x/\mu)+\theta(x-\mu t)\right.\\&\times\int_{x-\mu t}^{x}dx'e^{-2ax'/\mu}F\left(x',t-\frac{x-x'}{\mu}\right)\\&+\theta(t-x/\mu)\left[\int_0^x dx'e^{-2ax'/\mu}F\left(x',t-\frac{x-x'}{\mu}\right)\right.\\&\left.\left.+\int_0^{L-x}dx'F(x',t-x/\mu)\right]\right\}.\end{aligned} \tag{45}$$

The solution to this equation can be obtained by using the method of successive approximations [18]. Thus we write the solution as a power series of the smallness parameter ϵ as

$$F(x,t)=\sum_{m=0}^{\infty}\epsilon^mF_m(x,t). \tag{46}$$

Although the functions $\rho(x,t)$ and $N(t)$ appearing in Eq. (45) depend implicitly on ϵ, we will assume in expansion (46) that they are known functions of time and will ignore their ϵ dependence.

Substitution of Eq. (46) into Eq. (45) yields the following recursive set of equations for the coefficients of the different powers of ϵ:

$$F_0(x,t)=\rho(x,t)\{(e^{at}-1)\,\theta(x-\mu t)+(e^{ax/\mu}-1)\,\theta(\mu t-x)\}+\frac{1}{\mu}\,e^{2ax/\mu}I(t-x/\mu)\,\theta(t-x/\mu) \tag{47}$$

and, for $m\geqslant 1$,

$$F_m(x,t)=\frac{e^{2ax/\mu}}{L}\left\{N(t-x/\mu)\,\theta(t-x/\mu)\,\delta_{m,1}+\theta(x-\mu t)\times\int_{x-\mu t}^{x}dx'\,e^{-2ax'/\mu}F_{m-1}\left(x',t-\frac{x-x'}{\mu}\right)+\theta(t-x/\mu)\times\int_0^x dx'\,e^{-2ax'/\mu}F_{m-1}\left(x',t-\frac{x-x'}{\mu}\right)+\int_0^{L-x}dx'\,F_{m-1}(x',t-x/\mu)\right]. \tag{48}$$

Similarly to the case in which there is no external illumination source [1], the above expansion is found to converge when

$$|\epsilon|<\frac{1}{2}\left[\frac{\mu}{4aL}\left(\frac{\mu}{4aL}\,(e^{4aL/\mu}-1)-1\right)\right]^{-1/2}. \tag{49}$$

Finally, the variance of the total electron population obtained from Eqs. (14), (33), and (46) can be also expressed in powers of the smallness parameter

$$\sigma^2(t)=\sum_{m=0}^{\infty}\epsilon^m\sigma_m^2(t), \tag{50}$$

where

$$\sigma_m^2(t)\equiv\int_0^L dx\;F_m(x,t). \tag{51}$$

In particular, for a constant external illumination $I(t)=I_0>0$, the first two-expansion terms in Eq. (50) are

$$\sigma_0^2(t)=\theta(L-\mu t)\left\{(e^{at}-1)\int_{\mu t}^{L}dx\;\rho(x,t)+\int_0^{\mu t}dx\;\rho(x,t)\times(e^{ax/\mu}-1)+\frac{I_0}{2a}\,(e^{2at}-1)\right\}+\theta(\mu t-L)\times\left\{\int_0^L dx\;\rho(x,t)(e^{ax/\mu}-1)+\frac{I_0}{2a}\,(e^{2aL/\mu}-1)\right\} \tag{52}$$

and

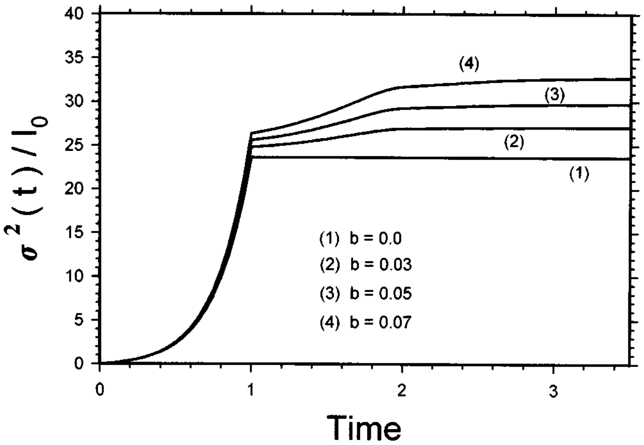

FIG. 3. Variance of the total-electron population as a function of time, as obtained from Eq. (50), keeping terms up to order ϵ. A constant illumination source $I(t)=I_0$ was used, with a primary production parameter $a=2$. In all the plots, the secondary production parameter b satisfies the convergence criterion given in Eq. (49). Dimensionless quantities are used, as defined in the text.

$$\sigma_1^2(t)=\theta(L-\mu t)\left\{2\left[\mu t e^{at}-\frac{\mu}{a}\,(e^{at}-1)\right]\int_{\mu t}^{L}dx\;\rho(x,t)+2\int_0^{\mu t}dx\;\rho(x,t)\left[xe^{ax/\mu}-\frac{\mu}{a}\,(e^{ax/\mu}-1)\right]+I_0\left[\frac{\mu t}{a}\,e^{2at}-\frac{\mu}{2a^2}\,(e^{2at}-1)\right]+\frac{\mu}{b}\int_0^{\mu t}dx\;\rho(x,t)e^{ax/\mu}-\frac{I_0}{b}\,\frac{\mu}{2a}\,(e^{2at}-1)\right\}+\theta(\mu t-L)\left\{2\int_0^L dx\;\rho(x,t)\left[xe^{ax/\mu}-\frac{\mu}{a}\,(e^{ax/\mu}-1)\right]+I_0\left[\frac{L}{a}\,e^{2aL/\mu}-\frac{\mu}{2a^2}\,(e^{2aL/\mu}-1)\right]+\frac{\mu}{b}\int_0^L dx\;\rho(x,t)e^{ax/\mu}-\frac{I_0}{b}\,\frac{\mu}{2a}\,(e^{2aL/\mu}-1)\right\}. \tag{53}$$

In order to illustrate the above results, in Figs. 3–6 we consider a subcritical electrical discharge, with a constant illumination source, i.e., $I(t)=I_0$. Similarly to Figs. 1 and 2, we take $N(0)=0$ and $a=2$, with all the variables in the dimensionless form defined in Sec. III. In Fig. 3 we show the time behavior of the variance in Eq. (50), up to order ϵ, i.e., $\sigma^2(t)\cong\sigma_0^2(t)+\epsilon\sigma_1^2(t)$. Several values are used for the secondary production parameter b satisfying the convergence criterion stated in Eq. (49), which for the case $a=2$ becomes $b<0.073$. Fast growth of the fluctuations is observed during the first transit period, approaching quickly their asymptotic values afterward. We note that the asymptotic value of the variance grows monotonically with the parameter b. This result is in good agreement with the exact asymptotic values of the variance, as obtained from Eq. (43), which are plotted in Fig. 4; a monotonically increasing behavior with the sec-

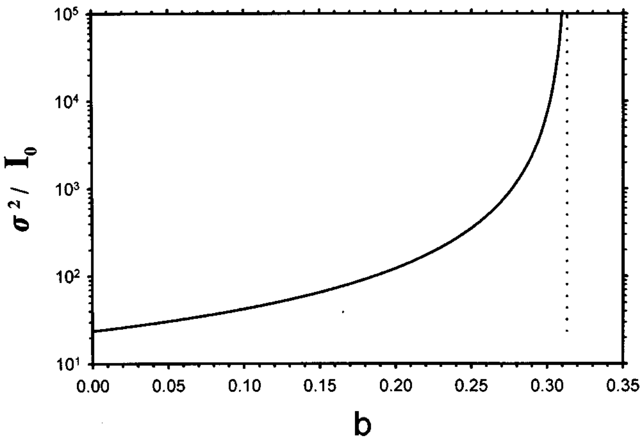

FIG. 4. Exact asymptotic value of the variance (43) as a function of the secondary production parameter $b < b_{\rm crit} = 0.313\,04$ $(a=2)$. Dimensionless quantities are used, as defined in the text.

ondary parameter b is observed here. It is shown in Appendix B that the asymptotic value of the variance diverges when b approaches $b_{\rm crit}=0.313\,04$ $(a=2)$ as

$$\sigma_\infty^2 \approx \frac{(1.210\,15) I_0}{(b-0.313\,04)^2}. \tag{54}$$

In Fig. 5 we plot the time behavior of the relative fluctuations $\sigma(t)/N(t)$ for the same discharge conditions as in Fig. 3. In all cases, the relative fluctuations are divergent at the early stages of the discharge since, in general, fluctuations are much more significant in small-size populations. The relative width decays in time and reaches its asymptotic value within a few transit times. The exact long-time values of the relative width as a function of the secondary production parameter b is shown in Fig. 6. This function decays monotonically with b, reaching its minimum value at $b = b_{\rm crit}$, given by

$$\left.\frac{\sigma_\infty}{N_\infty}\right|_{b=0.313\,04} = \frac{1.100\,07}{\sqrt{I_0}}, \tag{55}$$

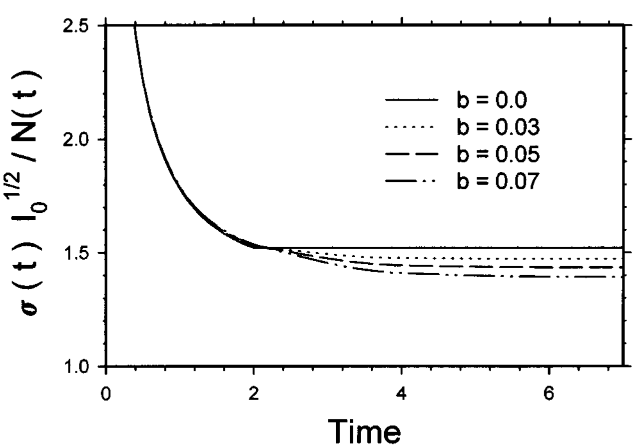

FIG. 5. Relative width of the fluctuations for the subcritical conditions in Fig. 3.

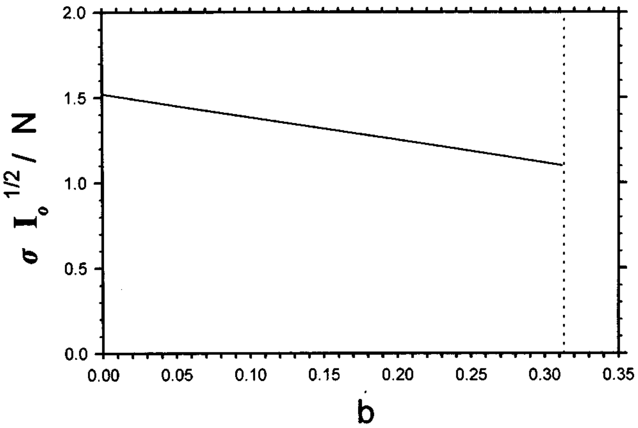

FIG. 6. Exact asymptotic value of the relative width of the fluctuations as a function of the secondary production parameter b for the subcritical conditions in Fig. 4.

as shown in Appendix B. We note that the variance-to-mean ratio is discontinuous at $b=b_{\rm crit}$; in fact, $\sigma_\infty/N_\infty \to \infty$ whenever $b > b_{\rm crit}$. On the other hand, for small values of b, it can be shown that (see Appendix C)

$$\frac{\sigma_\infty}{N_\infty} = \frac{1}{\sqrt{I_0}}\{1.5209 - 1.4307b + 0.7554b^2 + O(b^3)\}, \tag{56}$$

reflecting the quasilinear behavior in Fig. 6. It is worth noting that $\sigma_\infty/N_\infty \to \infty$ when $I_0 \to 0$ for critical as well as subcritical discharges, as follows from Eqs. (55) and (56), in agreement with previous results [1].

V. CONCLUDING REMARKS

In this work a characteristic functional approach was used to study fluctuations in the population of electrons within the plates of a discharge gap, in the presence of external illumination at the cathode. A Markov description of the electron population was developed by means of a space-discrete model. An equation for the characteristic functional associated with the corresponding probability density was obtained in the limit to the continuum, whose first- and second-order functional derivatives yielded time-evolution equations for the average total population and the density-density correlation function, respectively. The solutions to these equations were analyzed for several different discharge parameters.

For electrical discharges in the presence of a constant external illumination source, the results reveal the existence of a finite stationary population in the case of subcritical discharges only; a physical interpretation is provided for the corresponding steady-state solution, Eq. (28). In critical and supercritical discharges the population is found to diverge. On the other hand, the analysis of the fluctuations in the total population yields a nonvanishing finite stationary solution only in the case of subcritical discharges. It is found that the stationary electron variance grows monotonically as the parameters of the subcritical discharge approach the critical condition given by Eq. (24) and it becomes infinite for critical discharges. However, the corresponding variance-to-mean ratio σ/N shows a quasilinear behavior as a function of the secondary parameter b, for a fixed value of the primary production parameter a; the limit of σ/N when $b\to b_{\rm crit}^{-}$ exists; it is finite, but becomes infinite whenever the values of b are greater than $b_{\rm crit}$.

The results for the case where the external illumination vanishes at long times are in agreement with our previous work: A stationary average population is attained only for critical discharges; however, both the corresponding variance and variance-to-mean ratio diverge. It is concluded that stochastically stable solutions exist only for subcritical discharges with a nonvanishing external illumination source.

ACKNOWLEDGMENT

J.V. wishes to gratefully acknowledge partial financial support from UNAM Project No. DGAPA-IN101696.

APPENDIX A

In this appendix we show that for a critical discharge with a constant external illumination source the total electron population grows linearly in time for large values of t [19]. In order to simplify the notation, in this and the subsequent appendixes, all quantities will be expressed in the dimensionless form defined in the text.

For our purpose, we take Eq. (16) with $\rho(x,0)=0$ and assume that there is a constant illumination source of the type $I(t)=I_0\theta(t)$, where $\theta(t)$ is the Heaviside step function defined in Eq. (20); thus

$$\widetilde{N}(s)=\frac{I_0}{s}\frac{1-e^{-(s-a)}}{(s-a-b)\left[1+\dfrac{b}{s-a-b}\,e^{-(s-a)}\right]}. \quad \text{(A1)}$$

According to Eq. (24), in the case of a critical discharge we can write $b=a/(e^a-1)$, so that Eq. (A1) becomes

$$\widetilde{N}(s)=\frac{I_0}{s}\frac{1-e^{-(s-a)}}{(s-a)-\dfrac{a}{e^a-1}\,[1-e^{-(s-a)}]}. \quad \text{(A2)}$$

For small magnitude of the variable s, we can expand $\widetilde{N}(s)$ in powers of s by using the expansion

$$e^{-(s-a)}=e^a\sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}; \quad \text{(A3)}$$

after a few algebraic steps we obtain

$$\widetilde{N}(s)=\frac{I_0}{s^2}\frac{(1-e^a)}{[1+(a-1)e^a]}\left[1-e^a\sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}\right]\times\left[1-\frac{ae^a}{[1+(a-1)e^a]}\sum_{n=1}^{\infty}\frac{(-1)^n s^n}{(n+1)!}\right]^{-1}. \quad \text{(A4)}$$

Hence

$$\widetilde{N}(s)=\frac{1}{s^2}\left(\frac{I_0(e^a-1)^2}{1+(a-1)e^a}\right)-\frac{1}{s}\left(\frac{I_0e^a(e^a-1)[(a-2)e^a+a+2]}{2[1+(a-1)e^a]^2}\right)+a_0+a_1s+a_2s^2+\cdots, \quad \text{(A5)}$$

where $a_0,a_1,a_2,\ldots$ are real constants, whose explicit expressions are not relevant here. The leading term in Eq. (A5) is associated with the linear growth of the average population $N(t)$, while the second term is just a constant. It follows then that the asymptotic behavior of the mean total population is given by

$$N(t)\to\left(\frac{I_0(e^a-1)^2}{1+(a-1)e^a}\right)t-\frac{I_0e^a(e^a-1)[(a-2)e^a+a+2]}{2[1+(a-1)e^a]^2}. \quad \text{(A6)}$$

APPENDIX B

In this appendix we find the behavior of the long-time variance σ_∞^2 in Eq. (43), as the secondary production parameter b approaches the critical value, i.e., $b\to b_{\rm crit}^{-}$. To this end, we use the expansion parameter

$$\eta^2\equiv b_{\rm crit}-b, \quad \text{(B1)}$$

valid in the interval $0\leqslant b<b_{\rm crit}$, i.e., for subcritical discharges. In terms of the parameter η^2, the mean total electron population in Eq. (28) is written as

$$N_\infty=I_0\eta^{-2}; \quad \text{(B2)}$$

similarly, the parameter κ^2 in Eq. (38) is expressed as

$$\kappa^2=2a^{-1}e^a\eta^2-a^{-2}(e^{2a}-1)\eta^4. \quad \text{(B3)}$$

Hence, in the limit of small values of η, the parameter κ is given by

$$\kappa=\sqrt{2a^{-1}e^a}\,\eta\left[1-\left(\frac{e^{2a}-1}{4ae^a}\right)\eta^2+O(\eta^4)\right]; \quad \text{(B4)}$$

it follows that, in this limit,

$$\frac{e^{(1\pm\kappa)a}-1}{(1\pm\kappa)a}=b_{\rm crit}^{-1}\{1\pm\gamma\eta+O(\eta^2)\}, \quad \text{(B5)}$$

where $\gamma\equiv\sqrt{2a^{-1}e^a}(b_{\rm crit}e^a-1)$.

The substitution of the above results into Eqs. (40) and (41) for the coefficients C_1 and C_2, respectively, yields the expressions

$$C_1,C_2=\frac{I_0}{\eta^4}\left(\frac{a}{2e^a}\right)\left(\frac{b_{\rm crit}e^a}{b_{\rm crit}e^a-1}\right)+O\left(\frac{1}{\eta^3}\right) \quad \text{(B6)}$$

for small values of η. On the other hand, the coefficient C_3 in Eq. (42) is exactly given by

$$C_3=-I_0b_{\rm crit}^2\eta^{-4}. \quad \text{(B7)}$$

We note that all coefficients C_1, C_2, C_3 diverge as η^{-4} when $\eta \to 0$. Therefore, it is straightforward to show that the variance in Eq. (43) behaves as

$$\sigma_\infty^2 = \frac{I_0}{\eta^4}\left\{\frac{a}{b_{\rm crit}e^a - 1} - b_{\rm crit}\right\} + O\left(\frac{1}{\eta^3}\right) \tag{B8}$$

for small values of η. Finally, it follows from Eqs. (B2) and (B8) that the relative fluctuations in this region can be written as

$$\frac{\sigma_\infty}{N_\infty} = \sqrt{\frac{1}{I_0}\left(\frac{a}{b_{\rm crit}e^a - 1} - b_{\rm crit}\right)} + O(\eta), \tag{B9}$$

so that

$$\lim_{\eta\to 0}\frac{\sigma_\infty}{N_\infty} = \sqrt{\frac{1}{I_0}\left(\frac{a}{b_{\rm crit}e^a - 1} - b_{\rm crit}\right)}. \tag{B10}$$

APPENDIX C

Here we derive the stationary value of the variance-to-mean ratio σ_∞ / N_∞, as follows from Eqs. (28) and (43), for small secondary production parameter b. We first note from Eq. (28) for N_∞, together with the definition (24) of $b_{\rm crit}$, that

$$bN_\infty + I_0 = \frac{I_0}{1 - b/b_{\rm crit}}; \tag{C1}$$

for $b \ll b_{\rm crit}$, it can be approximated by

$$bN_\infty + I_0 = I_0[1 + (b/b_{\rm crit}) + (b/b_{\rm crit})^2 + O(b^3)]. \tag{C2}$$

On the other hand, for small b, the parameter κ that appears in Eqs. (40)–(43) can be expanded as

$$\kappa = 1 + \left(\frac{1}{a}\right)b - \left(\frac{e^{2a}}{2a^2}\right)b^2 + \left(\frac{e^{2a}}{2a^3}\right)b^3 + O(b^4). \tag{C3}$$

Similarly to the steps followed in Appendix B, we substitute Eqs. (C2) and (C3) into Eqs. (40)–(43) to obtain the variance σ_∞^2 for small b, from which it follows that

$$\frac{\sigma_\infty}{N_\infty} = \left(\frac{ae^a}{I_0(e^a - 1)}\right)^{1/2}[1 + k_1 b + k_2 b^2 + O(b^3)], \tag{C4}$$

where

$$k_1 \equiv \frac{1 + 2ae^a - e^{2a}}{2a(e^a - 1)}, \tag{C5}$$

$$k_2 \equiv \frac{3 - (4a^2+2)e^a + (4a-4)e^{2a} - (4a-2)e^{3a} + e^{4a}}{8a^2(e^a - 1)^2}, \tag{C6}$$

and Eq. (C1) has been used.

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